

On a universal quantum invariant of 3-manifolds *

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Quantum invariants of 3-manifolds was originally proposed by Witten in [22]. They are given by

$$Z_k(M, G) = \int e^{\sqrt{-1}kCS(A)} \mathcal{D}A$$

which is a topological invariant of a 3-manifold M with a compact Lie group G and an integer k , where the integral, what we call Feynman path integral, is over all G connections A , and the Chern-Simons functional $CS(A)$ is given by

$$CS(A) = \frac{1}{4\pi} \int_M \text{Tr}(A \wedge dA + \frac{2}{3}A^3).$$

We call $Z_k(M, G)$ quantum G invariant of M .

By perturbation theory we expect the asymptotic formula of $Z_k(M, G)$ for large k limit, see [3, 4, 9]. The formula is given by a sum over flat connections ρ as

$$Z_k(M, G) \sim \sum_{\rho} e^{kCS(\rho)} \frac{k^{d_{\rho}}}{R(\rho)} \exp \left(\sum_{\Gamma} \left(\frac{2\pi\sqrt{-1}}{k} \right)^{d(\Gamma)} I_{\Gamma}(M, \rho) \right).$$

Here d_{ρ} is the dimension of the cohomology group of the adjoint local system given by ρ and $R(\rho)$ is Reidemeister torsion, and the second sum is over all trivalent graphs Γ , and we mean by $d(\Gamma)$ half the number of vertices of Γ , and $I_{\Gamma}(M, \rho)$ is defined by an integral of a product of certain \mathfrak{g} valued 2-forms where the \mathfrak{g} bundle is twisted by ρ .

Here we restrict our attention to the contribution of the trivial connection ρ_0 in the asymptotic formula. Since the \mathfrak{g} bundle is trivial in this case, $I_{\Gamma}(M, \rho_0)$ is decomposed into a product of terms depending on \mathfrak{g} and M respectively as

$$I_{\Gamma}(M, \rho_0) = \Gamma(\mathfrak{g}) Z_{\Gamma}(M)$$

where $\Gamma(\mathfrak{g})$ is the value obtained by “substitute” \mathfrak{g} into Γ ; for the method of substitution, see for example [6]. We note that $\Gamma(\mathfrak{g})$ depends only on the equivalence class of Γ with respect to the AS and IHX relations. Further $Z_{\Gamma}(M)$ is given by

$$Z_{\Gamma}(M) = \int_{(x_1, \dots, x_{2d}) \in M^{2d} - \Delta} \prod_{\text{edges of } \Gamma} \omega(x_{l_i}, x_{r_i})$$

with a certain two form ω on $M \times M$, where we put $d = d(\Gamma)$, and the integral is over the product of $2d$ copies of M removed by its diagonal set Δ , and we take the product over all edges of Γ ; we associate the vertices of Γ with x_1, x_2, \dots, x_{2d} and we denote by x_{l_i} and x_{r_i} two ends of the i -th edges.

On the other hand we have a rigorous construction of quantum G invariants; we can obtain them by taking a sum of quantum (\mathfrak{g}, R) invariants of links over a certain set of representations R of \mathfrak{g} ; for sl_2 case, see [17]. Though we fix the parameter q in quantum invariants of links to be a root of unity in the construction, we can expect the asymptotic formula of quantum invariants of 3-manifolds; see [16] for an approach to the asymptotic formula for quantum $SO(3)$ invariants. For $SU(2)$ case, relations between the formula obtained in [16] and the trivial connection contribution of the above perturbative expansion are discussed in [18, 19].

Further we have the universal Vassiliev-Kontsevich invariant of links. It belongs to the space of chord diagrams consisting of solid and dashed lines subject to the AS, STU and IHX relations. The invariant is universal in the sense that it depends on neither a Lie algebra nor its representation and we can recover the quantum (\mathfrak{g}, R) invariant by “substituting” the Lie algebra \mathfrak{g} and the representation R to dashed and solid lines of the chord diagrams respectively.

Now we expect a rigorous construction of an invariant of 3-manifolds derived from the universal Vassiliev-Kontsevich invariant, which is universal in the sense that it consists of dashed chord diagrams and one should recover the trivial connection expansion of the perturbative expansion of quantum G invariant by substituting the Lie algebra of G into the dashed lines. In the present paper we get an invariant of 3-manifold from the universal Vassiliev-Kontsevich invariant, which is an infinite linear sum of chord diagrams, that is, it belongs to the space of chord diagrams consisting of dashed lines (i.e. trivalent graphs) subject to the AS and IHX relations. The space is graded by the number of vertices. The simplest one is the theta curve and we show that the coefficient of it corresponds to the Casson invariant.

In Section 1 we review a definition of the universal Vassiliev-Kontsevich invariant and show properties of it. In Section 2 we give a map to remove solid lines, and we define a universal quantum invariant using the map in Sections 3 and 4. We show some properties of the invariant in Section 5.

In the previous papers [12, 13] only the coefficient of the theta curve was obtained from the universal Vassiliev-Kontsevich invariant. The results of the present paper were announced in [14].

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1. MODIFIED UNIVERSAL VASSILIEV-KONTSEVICH INVARIANT

We will review a construction of the universal Vassiliev-Kontsevich invariant of framed oriented links, and the definition of modified one given in [11]. The modified invariant is well behaved under Kirby move II. We also show other properties of the invariant in this section.

1.1. Chord diagrams. A *uni-trivalent graph* is a graph every vertex of which is either univalent or trivalent. A uni-trivalent graph is *oriented* if at each trivalent vertex a cyclic order of edges is fixed.

Let X be a compact oriented 1-dimensional manifold whose components are labeled. A *chord diagram* with support X is the manifold X together with an oriented uni-trivalent graph whose univalent vertices are on X ; and the graph does not have any connected component homeomorphic to a circle. Note that our definition of a chord diagram is more general than that of [6, 10]. In Figures components of X are depicted by solid lines, while the graph is depicted by dashed lines. There may be connected components of the dashed graph which do not have univalent vertices. Each chord diagram has a natural topology. Two chord diagrams D, D' on X are regarded as equal if there is a homeomorphism $f : D \rightarrow D'$ such that $f|_X$ is a homeomorphism of X which preserves components and orientation.

Let $\mathcal{A}(X)$ be the vector space over \mathbb{C} spanned by chord diagrams with support X , subject to the AS, IHX and STU relations shown in Figure 1. The *degree* of a chord diagram is half the number of vertices of the dashed graph. Since the relations AS, IHX and STU respect the degree, there is a grading on $\mathcal{A}(X)$ induced by this degree. We denote by $\hat{\mathcal{A}}(X)$ the completion of $\mathcal{A}(X)$ with respect to the degree. Note that each map on $\mathcal{A}(X)$ defined in the following has the natural extension of it on $\hat{\mathcal{A}}(X)$, and that we use the same notation for the corresponding maps on $\mathcal{A}(X)$ and $\hat{\mathcal{A}}(X)$.

Suppose C is a component of X . Reversing the orientation of C , from X we get X' . Let $S_{(C)} : \mathcal{A}(X) \rightarrow \mathcal{A}(X')$ be the linear mapping which transfers every chord diagram D in $\mathcal{A}(X)$ to $S_{(C)}(D)$ obtained from D by reversing the orientation of C and multiplying by $(-1)^m$, where m is the number of vertices of the dashed graph on the component C .

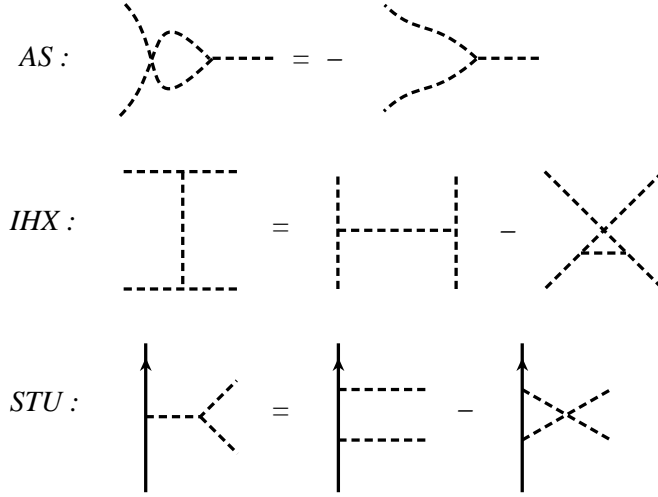


FIGURE 1. The AS, IHX and STU relations

Replacing C by 2 copies of C , from X we get $X^{(2,C)}$, with a projection $p : X^{(2,C)} \rightarrow X$. If x is a point on C then $p^{-1}(x)$ consists of 2 points, while if x is a point of other components, then $p^{-1}(x)$ consists of one point. Let D be a chord diagram on X , with the dashed graph G . Suppose that there are m univalent vertices of G on C . Consider all possible new chord diagrams on $X^{(2,C)}$ with the same dashed graph G such that if a univalent vertex of G is attached to a point x on X in D , then this vertex is attached to a point in $p^{-1}(x)$ in the new chord diagram. There are 2^m such chord diagrams, and their sum is denoted by $\Delta_C(D)$. It is easy to check that these linear mappings $S_{(C)}$ and $\Delta_{(C)}$ are well-defined on $\mathcal{A}(X)$, and naturally extended to the maps on $\hat{\mathcal{A}}(X)$.

Suppose that X and X' have distinguished components C and C' respectively, and that X consists of loop components only. Let $D \in \mathcal{A}(X)$ and $D' \in \mathcal{A}(X')$ be two chord diagrams. From each of C and C' we remove a small arc which does not contain any vertices. The remaining part of C is an arc which we glue to C' in the place of the removed arc such that the orientations are compatible. The new chord diagram is called the *connected sum of D and D' along the distinguished components*; it does not depend on the locations of the removed arcs, which follows from the STU relation and the fact that all components of X are loops. The proof is the same as in the case $X = X' = S^1$ as in [6].

We define a co-multiplication $\hat{\Delta}$ in $\mathcal{A}(X)$ and $\hat{\mathcal{A}}(X)$ as follows. A *chord sub-diagram* of a chord diagram D with dashed graph G is any chord diagram obtained from D by

removing some connected components of G . The *complement chord sub-diagram* of a chord sub-diagram D' is the chord sub-diagram obtained by removing components of G which are in D' . We define

$$\hat{\Delta}(D) = \sum D' \otimes D''.$$

Here the sum is over all chord sub-diagrams D' of D , and D'' is the complement of D' . This co-multiplication is co-commutative.

1.2. Associator. Let $\mathbb{C} \langle\langle A, B \rangle\rangle$ be the algebra over \mathbb{C} of all formal power series in two non-commutative symbols A, B . We are going to define an element $\varphi \in \mathbb{C} \langle\langle A, B \rangle\rangle$, known as the Drinfeld associator.

We put

$$\zeta(i_1, \dots, i_k) = \sum_{n_1 < \dots < n_k \in \mathbb{N}} \frac{1}{n_1^{i_1} \dots n_k^{i_k}}, \quad (1.1)$$

for natural numbers i_1, \dots, i_k satisfying $i_k \geq 2$. These values, called multiple zeta values, have recently gained much attention among number theorists. In what follows bold letters $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}$ stand for non-negative multi-indices. For a multi-index $\mathbf{p} = (p_1, \dots, p_k)$, we call k the *length* of \mathbf{p} . Let $\mathbf{1}_k$ be the multi-index consisting of k letters 1. We denote $\sum p_i$ by $|\mathbf{p}|$. For two multi-indices \mathbf{p} and \mathbf{q} of the same length k , we put $\eta(\mathbf{p}; \mathbf{q}) = 0$ if one of p_i, q_i is 0,

$$\eta(\mathbf{p}; \mathbf{q}) = \zeta(\mathbf{1}_{p_1-1}, q_1 + 1, \mathbf{1}_{p_2-1}, q_2 + 1, \dots, \mathbf{1}_{p_k-1}, q_k + 1),$$

otherwise. Further we set two notations by

$$(A, B)^{(\mathbf{p}, \mathbf{q})} = A^{p_1} B^{q_1} A^{p_2} B^{q_2} \dots A^{p_k} B^{q_k},$$

$$\binom{\mathbf{p}}{\mathbf{q}} = \binom{p_1}{q_1} \binom{p_2}{q_2} \dots \binom{p_k}{q_k}.$$

Using the above notations we define the *associator* φ by

$$\varphi(A, B) = 1 + \sum_{\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s} \geq \mathbf{0}} (-1)^{|\mathbf{r}| + |\mathbf{q}|} \eta(\mathbf{p} + \mathbf{r}; \mathbf{q} + \mathbf{s}) \binom{\mathbf{p} + \mathbf{r}}{\mathbf{r}} \binom{\mathbf{q} + \mathbf{s}}{\mathbf{s}} B^{|\mathbf{s}|} (A, B)^{(\mathbf{p}, \mathbf{q})} A^{|\mathbf{r}|}.$$

Here the sum is over all multi-indices $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}$ of the same length k where $k = 1, 2, 3, \dots$. Note that there exists the inverse of $\varphi(A, B)$; we denote it by $\varphi^{-1}(A, B)$.

1.3. The universal Vassiliev-Kontsevich invariant for framed oriented links.

In this subsection, we will define the universal Vassiliev-Kontsevich invariant $\hat{Z}(L)$ of a framed oriented link L . Before defining it, we define an invariant $Z(\mathcal{D})$ of a link diagram \mathcal{D} .

Suppose \mathcal{D} is an oriented l -component link diagram in a plane with a fixed coordinate system. Using horizontal lines one can decompose \mathcal{D} into elementary tangle diagrams. Here we mean an *elementary tangle diagram* is one of the tangle diagrams shown in Figure 2, maybe with reverse orientation on some components. We number the components of the elementary tangle diagrams from left to right. There are n components in $X_{k,n}^\pm$, and the crossing is on the k -th and $(k+1)$ -th components. There are $n-1$ components in $U_{k,n}, V_{k,n}$, and the non-straight component is numbered by k . We define $Z(\mathcal{D}) \in \hat{\mathcal{A}}(\coprod^l S^1)$ as follows.

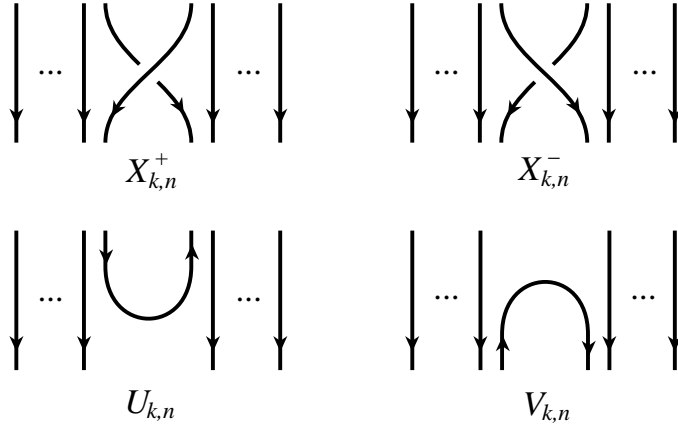


FIGURE 2. Elementary tangle diagrams

Firstly we define $Z(T)$ for each elementary tangle diagram T . When X is n vertical straight numbered lines with downward orientations, $\hat{\mathcal{A}}(X)$ is denoted by \mathcal{P}_n . All the \mathcal{P}_n are algebras: the product of two chord diagrams D_1 and D_2 is obtained by placing D_1 on the top of D_2 . The algebra \mathcal{P}_1 is commutative (see [6, 8]). Consider the following element $x_{k,n}$ in \mathcal{P}_n

$$x_{k,n}^+ = \varphi^{-1}\left(\frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{k-1} \Omega_{i,k}, \frac{\Omega_{k,k+1}}{2\pi\sqrt{-1}}\right) \exp\left(\frac{\Omega_{k,k+1}}{2}\right) \varphi\left(\frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{k-1} \Omega_{i,k+1}, \frac{\Omega_{k,k+1}}{2\pi\sqrt{-1}}\right).$$

Here φ is defined in the previous subsection, and $\Omega_{i,j}$ is the chord diagram in \mathcal{P}_n with the dashed graph being a line connecting the i -th and the j -th string. Similarly we put

$$\begin{aligned} x_{k,n}^- &= \varphi^{-1}\left(\frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{k-1} \Omega_{i,k}, \frac{\Omega_{k,k+1}}{2\pi\sqrt{-1}}\right) \exp\left(-\frac{\Omega_{k,k+1}}{2}\right) \varphi\left(\frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{k-1} \Omega_{i,k+1}, \frac{\Omega_{k,k+1}}{2\pi\sqrt{-1}}\right), \\ u_{k,n} &= \varphi^{-1}\left(\frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{k-1} \Omega_{i,k}, \frac{\Omega_{k,k+1}}{2\pi\sqrt{-1}}\right), \\ v_{k,n} &= \varphi\left(\frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{k-1} \Omega_{i,k}, \frac{\Omega_{k,k+1}}{2\pi\sqrt{-1}}\right). \end{aligned}$$

We define $Z(X_{k,n}^+)$ as the element in $\hat{\mathcal{A}}(X_{k,n}^+)$ obtained by placing $X_{k,n}^+$ without any dashed graph on the top of $x_{k,n}^+$. Here we also use the notation $X_{k,n}^+$ for the set of solid lines of $X_{k,n}^+$. Similarly, $Z(X_{k,n}^-)$ is the element in $\hat{\mathcal{A}}(X_{k,n}^-)$ obtained by placing $X_{k,n}^-$ on the top of $x_{k,n}^-$. Further $Z(U_{k,n})$ is obtained by placing $U_{k,n}$ on the bottom of $S_{C_{k+1}}(u_{k,n})$, where C_{k+1} is the $(k+1)$ -th strings of the support of chord diagrams in \mathcal{P}_n , and S_{C_k} is defined in the previous subsection. Furthermore $Z(V_{k,n})$ is obtained by placing $V_{k,n}$ on the top of $S_{C_k}(v_{k,n})$. We also show pictures of the definition in Figure 3.

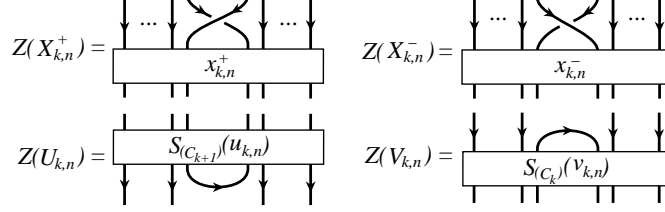


FIGURE 3. The definition of $Z(T)$ for each elementary tangle diagram T

Secondly we define $Z(T)$ for each elementary tangle diagram T with arbitrary orientation by using $Z(T) = S_C(Z(T'))$ some times, where T is obtained from T' by reversing the orientation of a component C .

Lastly we define $Z(\mathcal{D}) \in \hat{\mathcal{A}}(\coprod^l S^1)$ for an oriented l -component link diagram \mathcal{D} . We can decompose \mathcal{D} into elementary tangle diagrams T_1, T_2, \dots, T_m by horizontal lines, counting from top to bottom. We set $Z(\mathcal{D}) = Z(T_1) \times Z(T_2) \times \dots \times Z(T_m)$. Here, for chord diagrams D_i composing $Z(T_i)$ ($i = 1, 2, \dots, m$), we mean by $D_1 \times \dots \times D_m$ the chord diagram on $\coprod^l S^1$ obtained by placing D_1 on the top of D_2 , and placing the union on top of D_3 , and so on. Note that the supports of D_1, D_2, \dots, D_m can be glued together, and the result consists of l solid loops. It is known that $Z(\mathcal{D})$ is defined uniquely for an oriented link diagram \mathcal{D} , not depending on the decomposition of \mathcal{D} , i.e. more precisely, not depending

on an isotopy of the link diagram \mathcal{D} in the plane which preserves the maximal points of \mathcal{D} .

Suppose now that L is a framed oriented link, represented by a link diagram \mathcal{D} with blackboard framing. Further suppose that the i -th components of \mathcal{D} has s_i maximal points with respect to the height function of the plane. We define an invariant $\hat{Z}(L)$ by

$$\hat{Z}(L) = Z(\mathcal{D}) \# (\nu^{s_1} \otimes \cdots \otimes \nu^{s_l}) \in \hat{\mathcal{A}}(\prod_{i=1}^l S^1),$$

where we put $\nu = Z(U)^{-1}$ for the link diagram U shown in Figure 4, and we mean by the above formula that $\hat{Z}(L)$ is obtained from $Z(\mathcal{D})$ by successively taking connected sum with ν^{s_i} along the i -th component. (Note that a notation $\hat{Z}_f(L)$ was used instead of $\hat{Z}(L)$ in the previous papers [10, 11].) The invariant $\hat{Z}(L)$ of a framed link L is well-defined, not depending on the choice of its link diagram \mathcal{D} , see [10]. Note that, for the trivial knot K with framing 0, $\hat{Z}(K)$ is not trivial; in fact, we have $\hat{Z}(K) = \nu$.

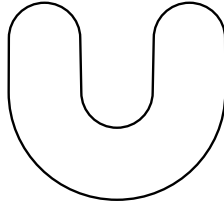


FIGURE 4. The link diagram U

The invariant $\hat{Z}(L)$ is an invariant of framed oriented links such that it contains in itself all Vassiliev invariants of framed oriented links, and it is a generalization of the Kontsevich integral. We call this invariant *the universal Vassiliev-Kontsevich invariant*.

1.4. Parallel of framed links. The following was proved in [11].

Proposition 1.1. *Let C be a component of an oriented framed link L .*

1. *Let $L^{(2,C)}$ be the link obtained from L by replacing C by two push-offs of C using the frame. Then we have the following formula;*

$$\hat{Z}(L^{(2,C)}) = \Delta_{(C)}(\hat{Z}(L)). \quad (1.2)$$

2. *Let L' be obtained from L by reversing the orientation of C . Then we have the following formula;*

$$\hat{Z}(L') = S_{(C)}(\hat{Z}(L)). \quad (1.3)$$

Now we put

$$\check{Z}(L) = \hat{Z}(L) \# (\nu \otimes \cdots \otimes \nu) \in \hat{\mathcal{A}}(\coprod^l S^1).$$

This formula means that $\check{Z}(L)$ is obtained from $\hat{Z}(L)$ by successively taking connected sum with ν along every component of L . It is easy to see that Proposition 1.1 is also valid if we replace \hat{Z} by \check{Z} in the statement of the proposition.

Suppose that X consists of n components C_1, \dots, C_n . We denote by $X \sqcup X$ the disjoint union of two copies of X . We define a linear map:

$$p : \hat{\mathcal{A}}(X \sqcup X) \rightarrow \hat{\mathcal{A}}(X)^{\otimes 2}$$

as follows. If D is a chord diagram having dashed graph connecting the two copies of X , then we put $p(D) = 0$. Otherwise, D splits into a disjoint union of two chord diagrams D_1 and D_2 on the first and the second copies of X respectively, and we put $p(D) = D_1 \otimes D_2$.

Then by definition we have

$$p \circ \Delta_{(C_1, \dots, C_n)}(D) = \hat{\Delta}(D), \quad (1.4)$$

where $\Delta_{(C_1, \dots, C_n)} : \hat{\mathcal{A}}(X) \rightarrow \hat{\mathcal{A}}(X \sqcup X)$ is the mapping obtained by successively applying $\Delta_{(C_i)}, i = 1, \dots, n$.

Theorem 1.2. *Let L be an oriented framed link. Then the following formula holds;*

$$\hat{\Delta}(\check{Z}(L)) = \check{Z}(L) \otimes \check{Z}(L).$$

Proof. By the equations (1.2) and (1.4), the left hand side of the required formula is equal to $p(\check{Z}(L^{(2)}))$. Identifying $\hat{\mathcal{A}}(X) \otimes \hat{\mathcal{A}}(X)$ with a subset of $\hat{\mathcal{A}}(X \sqcup X)$ naturally, the right hand side of the required formula becomes equal to $\check{Z}(L \circ L)$, where $L \circ L$ is the split union of two copies of L , i.e., the disjoint union not winding with each other.

Hence it is sufficient to show that $p(\check{Z}(L^{(2)}))$ is equal to $\check{Z}(L \circ L)$. Note that we can obtain $L \circ L$ from $L^{(2)}$ by taking crossing changes between the first and the second copies of L . In this procedure $\check{Z}(L^{(2)})$ changes to $\check{Z}(L \circ L)$ only in terms which have dashed graphs connecting the two copies of X , since the difference of a crossing change is locally given by

$$\begin{aligned} x_{k,n}^+ - x_{k,n}^- &= \varphi^{-1}(\dots) \left(\exp\left(\frac{\Omega_{k,k+1}}{2}\right) - \exp\left(-\frac{\Omega_{k,k+1}}{2}\right) \right) \varphi(\dots) \\ &= \varphi^{-1}(\dots) (\Omega_{k,k+1} + \text{higher terms with respect to } \Omega_{k,k+1}) \varphi(\dots). \end{aligned}$$

Therefore, after the procedure, $\check{Z}(L^{(2)})$ becomes an element of $\hat{\mathcal{A}}(X \sqcup X)$ which is different from $\check{Z}(L^{(2)})$ only in terms with dashed graphs connecting the two copies of X . Further

the element should vanish in such terms, since it is equal to the invariant of a split union. Hence it must be equal to $p(\check{Z}(L^{(2)}))$; it is also equal to $\check{Z}(L \circ L)$. \square

1.5. The change under Kirby move II. We have the following proposition proved in [11]. By this proposition we can get the change of $\check{Z}(L)$ under Kirby move II. Throughout the present paper we mean by Kirby move II the handle slide move defined by Kirby in [7]; we show a picture of the move in Figure 5.

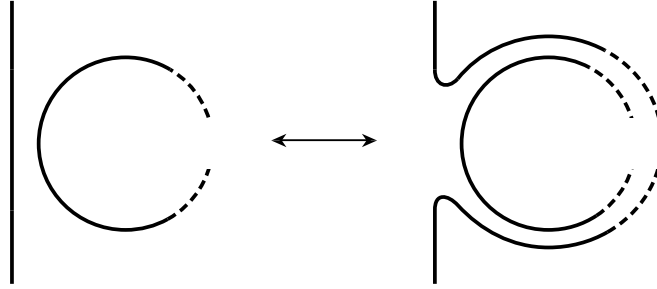


FIGURE 5. Kirby move II (the handle slide move)

Proposition 1.3 ([11]). *Let L be an oriented framed link, and L' a framed link obtained from L by Kirby move II which preserves the orientation. Then $\check{Z}(L')$ can be obtained from $\check{Z}(L)$ by replacing the left picture in Figure 6 with the right picture.*

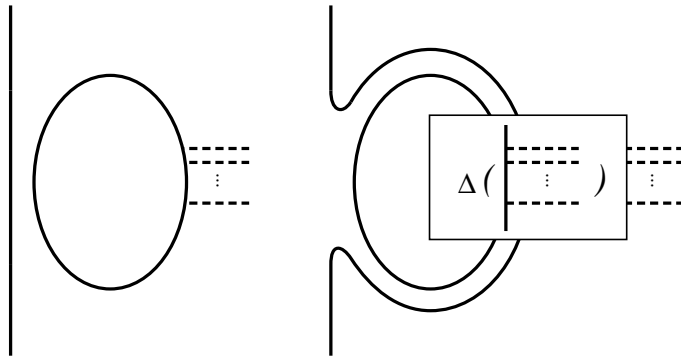


FIGURE 6. The change under Kirby move II

2. REPLACING SOLID CIRCLES BY DASHED LINES

Our aim of this section is to construct the series of the maps ι_n , whose definition is given in the end of this section. The universal Vassiliev-Kontsevich invariant belongs to

the space consisting of both of solid and dashed lines. When we consider quantum (\mathfrak{g}, R) invariants of links for a Lie algebra \mathfrak{g} and its representation R , the solid and dashed lines corresponds to R and \mathfrak{g} respectively. Further, we know that no particular representation R is specified in quantum \mathfrak{g} invariants of 3-manifolds. It implies that our obstruction in constructing invariants of 3-manifolds from the universal Vassiliev-Kontsevich invariant might be the existence of solid lines. We will remove solid lines from the space of chord diagrams by the maps ι_n when we construct invariants of 3-manifolds in the following section.

2.1. Chord diagrams with finite support. Let X be a finite set consisting of m ordered points named $0, 1, 2, \dots, m-1$. We consider chord diagrams with support X , that is, oriented uni-trivalent graphs whose m univalent vertices are on the m fixed points respectively. We denote by $\mathcal{A}(m)$ the vector space over \mathbb{C} spanned by such chord diagrams subject to the AS and IHX relations. Further we denote by $\mathcal{A}(m)_{\text{tree}}$ the vector subspace of $\mathcal{A}(m)$ spanned by connected and simply connected graphs; note that the AS and IHX relations are closed in the subspace.

For an element τ in the symmetric group \mathcal{S}_{m-2} acting on the set $\{1, 2, \dots, m-2\}$, let $T_\tau \in \mathcal{A}(m)$ be the graph shown in Figure 7.

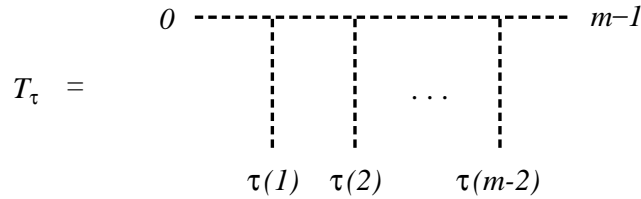


FIGURE 7. The definition of T_τ

Lemma 2.1. *We can take the set of T_τ as basis of the space $\mathcal{A}(m)_{\text{tree}}$; in particular the space is $(m-2)!$ dimensional.*

Proof. Let D be any chord diagram in $\mathcal{A}(m)_{\text{tree}}$. We put red color on the path connecting the two univalent vertices 0 and $m-1$. We can deform D into a linear sum of T_τ 's by induction on the number of trivalent vertices on the red path as follows. We choose a trivalent vertex next to the red path, and apply the IHX relation regarding the segment connecting the vertex and the red path as the character “I” in “IHX”. Then the number

of trivalent vertices on the red path increases. Hence we can show that the space $\mathcal{A}(m)_{\text{tree}}$ is spanned by the set of T_τ .

In order to complete the proof of this lemma, it is sufficient to prove that T_τ 's are linearly independent. Suppose that T_τ could be expressed as a linear sum of other T_σ 's. We can “substitute” a Lie algebra $sl(m, \mathbb{C})$ to dashed lines (see [6]) to make a linear map of $\mathcal{A}(m)_{\text{tree}}$ to \mathbb{C} . For $j < k$ let E_{jk} be the element in $sl(m, \mathbb{C})$ which has (j, k) entry 1 and the other entries 0. We have a relation $[E_{ij}, E_{jk}] = E_{ik}$. If we substitute E_{12} to the univalent vertex 0 and $E_{k+1, k+2}$ to the vertex $\tau(k)$ for $k = 1, 2, \dots, m-2$, then T_σ always vanishes unless $\sigma = \tau$, though T_τ does not vanish when we substitute the dual of E_{1m} to the vertex $m-1$, where we mean the dual with respect to the Killing form. This is a contradiction, completing the proof. \square

2.2. Chord diagrams behaving in a similar way as a solid circle. We define $T_m \in \mathcal{A}(m)_{\text{tree}}$ by

$$T_m = \sum_{\tau \in \mathcal{S}_{m-2}} \frac{(-1)^{r(\tau)}}{(m-1) \binom{m-2}{r(\tau)}} T_\tau,$$

where we denote by $r(\tau)$ the number of k which satisfies $\tau(k) > \tau(k+1)$. In this subsection, our aim is to show Proposition 2.7. We begin with the following lemma, which is a weak form of Proposition 2.7; we can interchange any two adjacent univalent vertices in Proposition 2.7 whereas we can do it for particular pairs in Lemma 2.2. We will show symmetries of T_m in this subsection to obtain Proposition 2.7 from Lemma 2.2.

Lemma 2.2. *If $1 \leq k \leq m-3$, then the difference between T_m and the chord diagram obtained from T_m by changing two univalent vertices k and $k+1$ can be expressed using T_{m-1} as shown in Figure 8.*

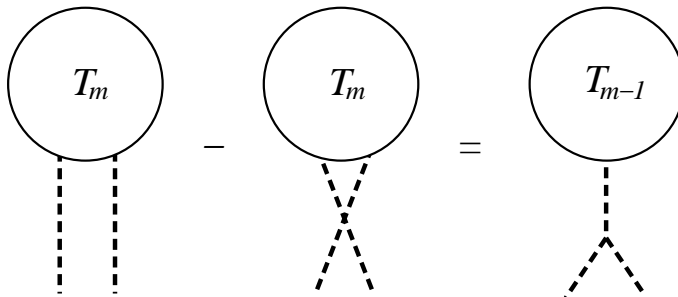


FIGURE 8. A property of T_m similar to the STU relation

Proof. Since we can take the set of T_τ as basis of $\mathcal{A}(m)_{\text{tree}}$, we can express both sides of the required formula as a linear sum of T_τ 's. It is sufficient to show that the coefficients of T_τ in both sides are equal for each τ .

If $|\tau^{-1}(k) - \tau^{-1}(k+1)| \geq 2$, then the coefficient of the left hand side is equal to zero, since $r(\tau) = r((k \ k+1) \circ \tau)$ holds in this case, where we mean by $(k \ k+1)$ the interchange of k and $k+1$. On the other hand, the coefficient of the right hand side is equal to zero, since the right hand side is equal to a linear sum of T_τ for τ satisfying $|\tau^{-1}(k) - \tau^{-1}(k+1)| = \pm 1$; we can see it by applying the IHX relation in the right hand side. Therefore the coefficients of T_τ in both sides are equal in this case.

If $\tau^{-1}(k) - \tau^{-1}(k+1) = -1$, then the coefficient of the left hand side is equal to $t_{m,\tau} - t_{m,(k \ k+1) \circ \tau}$ where we put

$$t_{m,\tau} = \frac{(-1)^{r(\tau)}}{(m-1) \binom{m-2}{r(\tau)}}.$$

In this case we have $r((k \ k+1) \circ \tau) = r(\tau) + 1$ by the definition of $r(\cdot)$. Hence we have

$$\begin{aligned} t_{m,\tau} - t_{m,(k \ k+1) \circ \tau} &= \frac{(-1)^{r(\tau)}}{(m-1) \binom{m-2}{r(\tau)}} - \frac{(-1)^{r(\tau)+1}}{(m-1) \binom{m-2}{r(\tau)+1}} \\ &= \frac{(-1)^{r(\tau)}(m-r-2)}{(m-1)(m-2) \binom{m-3}{r(\tau)}} - \frac{(-1)^{r(\tau)+1}(r+1)}{(m-1)(m-2) \binom{m-3}{r(\tau)}} = \frac{(-1)^{r(\tau)}}{(m-2) \binom{m-3}{r(\tau)}} \end{aligned}$$

On the other hand, the contribution of the right hand side to T_τ comes from $T_{\hat{\tau}} \in \mathcal{A}(m-1)$, where we define $\hat{\tau} \in \mathcal{S}_{m-1}$ by putting $\hat{\tau}^{-1}(j)$ to be $\tau^{-1}(j)$ if $j \leq k$, $\tau^{-1}(j+1) - 1$ if $j > k$. The coefficient of $T_{\hat{\tau}}$ is equal to $t_{m-1,\hat{\tau}}$; in this case we have $r(\hat{\tau}) = r(\tau)$ by the definition of $r(\cdot)$. Therefore the coefficients of both sides are equal, completing this case.

If $\tau^{-1}(k) - \tau^{-1}(k+1) = 1$, we can show that the coefficients are equal in a similar way as above, completing the proof. \square

Lemma 2.3. *The chord diagram T_m is symmetric with respect to mirror image which replaces $0, 1, \dots, m-1$ with $m-1, m-2, \dots, 0$ respectively, see Figure 13 for simple cases. To be exact, the inversion replaces T_m to $(-1)^{m-2}T_m$ because it changes the orientations of $m-2$ trivalent vertices.*

Proof. By the inversion which replaces $0, 1, \dots, m-1$ with $m-1, m-2, \dots, 0$, the chord diagram T_τ moves to $(-1)^{m-2}T_{\tau'}$ where

$$\tau' = \begin{pmatrix} 1 & 2 & \cdots & m-2 \\ m-2 & m-1 & \cdots & 1 \end{pmatrix} \circ \tau \circ \begin{pmatrix} 1 & 2 & \cdots & m-2 \\ m-2 & m-1 & \cdots & 1 \end{pmatrix}$$

and the sign is derived from the number of trivalent vertices; we use the AS relation such times. We can obtain $r(\tau) = r(\tau')$ by definition of $r(\cdot)$. Therefore the inversion maps T_m to $(-1)^{m-2}T_m$, completing the proof. \square

Lemma 2.4. *The chord diagram T_m is symmetric with respect to mirror image which replaces $0, 1, 2, \dots, m-2, m-1$ with $m-2, m-3, m-4, \dots, 0, m-1$ respectively. To be exact, the inversion replaces T_m to $(-1)^{m-2}T_m$ because of the orientations of trivalent vertices.*

Proof. Consider the linear map i of $\mathcal{A}(m)_{\text{tree}}$ to $\mathcal{A}(m+1)_{\text{tree}}$ which maps a chord diagram D to the diagram D added a small branch near the univalent vertex $m-1$ as shown in Figure 9. The map i is an injection; we can see it by taking basis of $\mathcal{A}(m)_{\text{tree}}$ and $\mathcal{A}(m+1)_{\text{tree}}$ in the way how we choose the red path connecting 0 and $m-1$ and connecting 0 and m respectively as in the proof of Lemma 2.1.

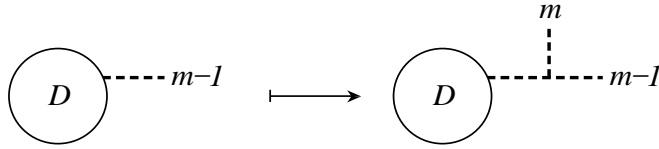


FIGURE 9. The map of $\mathcal{A}(m)_{\text{tree}}$ to $\mathcal{A}(m+1)_{\text{tree}}$

We denote by \mathcal{S}'_{m-1} the symmetric group acting on the set $\{0, 1, \dots, m-2\}$. For an element $\sigma \in \mathcal{S}'_{m-1}$, let $S_\sigma \in \mathcal{A}(m+1)_{\text{tree}}$ be the chord diagram shown in Figure 10. By Lemma 2.1 we can take the set of S_σ as basis of $\mathcal{A}(m+1)_{\text{tree}}$.

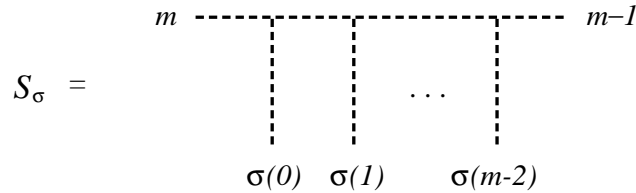


FIGURE 10. The definition of S_σ

We can express $i(T_m)$ as in Lemma 2.5 below. In order to complete the proof of the present lemma, it is sufficient to show that $i(T_m)$ is symmetric with respect to the inversion

replacing $0, 1, \dots, m-2$ with $m-2, m-1, \dots, 0$, since the map i is an injection. The inversion maps S_σ to $S_{\sigma'}$ where

$$\sigma' = \begin{pmatrix} 0 & 1 & \cdots & m-2 \\ m-2 & m-1 & \cdots & 0 \end{pmatrix} \circ \sigma,$$

note that we have no change of sign in this case because we fix the ends $m-1$ and m of the “red path”. We can obtain $r(\sigma') = m-2 - r(\sigma)$ by the definition of $r(\cdot)$. By Lemma 2.5 below, the inversion maps $i(T_m)$ to $(-1)^{m-2}i(T_m)$, completing the proof. \square

Lemma 2.5.

$$i(T_m) = \sum_{\sigma \in \mathcal{S}'_{m-1}} \frac{(-1)^{r(\sigma)}}{(m-1) \binom{m-2}{r(\sigma)}} S_\sigma.$$

Proof. We put $S_{m+1} = i(T_m)$. Since the set of S_σ is basis of $\mathcal{A}(m+1)_{\text{tree}}$, we can put $S_{m+1} = \sum_\sigma s_\sigma S_\sigma$ with some scalars s_σ . In the following of this proof we will show the following formula

$$s_\sigma = \frac{(-1)^{r(\sigma)}}{(m-1) \binom{m-2}{r(\sigma)}} \quad (2.1)$$

in two steps by induction on m .

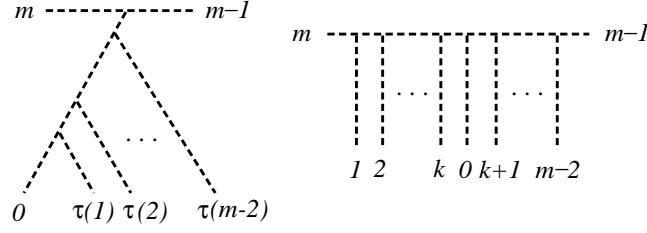
Step 1. If σ is a cyclic permutation $(0, 1, 2, \dots, k)$ for an integer k , we can obtain (2.1) by definition of $i(T_m)$ as follows. We consider that what T_τ contributes to such S_σ after changing basis of $\mathcal{A}(m)_{\text{tree}}$ to that of $\mathcal{A}(m+1)_{\text{tree}}$; we expand the left picture in Figure 11 using the IHX relation to obtain the right picture. We use the IHX relation replacing “I” with the difference of “H” and “X”. Noting that the univalent vertex 0 interchanges k times with another vertex, we must use “X” k times in the expansion. Hence the possibilities of τ are as follow;

$$\begin{aligned} \tau &= \begin{pmatrix} 1 & 2 & \cdots & l_k - 1 & l_k & l_k + 1 & \cdots & l_1 - 1 & l_1 & l_1 + 1 & \cdots & m-2 \\ k+1 & k+2 & \cdots & l_k + k - 1 & k & l_k + k & \cdots & l_1 & 1 & l_1 + 1 & \cdots & m-2 \end{pmatrix} \\ \text{or } &\begin{pmatrix} 1 & 2 & \cdots & l_{k-1} - 1 & l_{k-1} & l_{k-1} + 1 & \cdots & m-2 \\ k & k+1 & \cdots & l_{k-1} + k - 2 & k-1 & l_{k-1} + k - 1 & \cdots & m-2 \end{pmatrix} \end{aligned}$$

In the first type we can freely choose $\{l_k, l_{k-1}, \dots, l_1\}$ from $\{2, 3, \dots, m-2\}$. Hence there are $\binom{m-3}{k}$ possibilities of τ , and $r(\tau) = k$ holds in this type. Similarly we have $\binom{m-3}{k-1}$ possibilities of τ , which satisfy $r(\tau) = k-1$ in the second type. Therefore we have

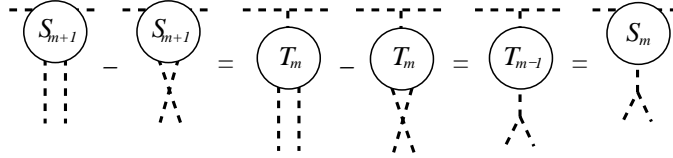
$$s_\sigma = (-1)^k \binom{m-3}{k} \frac{(-1)^k}{(m-1) \binom{m-2}{k}} + (-1)^k \binom{m-3}{k-1} \frac{(-1)^{k-1}}{(m-1) \binom{m-2}{k-1}}$$

where the term $(-1)^k$ is derived from the number of usage of “X”. This formula satisfies (2.1), completing Step 1.

FIGURE 11. Expanding T_τ

Step 2. In this step we will show that if (2.1) holds for σ then it also holds for $(k \ k+1) \circ \sigma$ for any $k = 1, 2, \dots, m-3$, where we mean by $(k \ k+1)$ the interchange of k and $k+1$.

We have the formula shown in Figure 12, where the first and third equalities in the figure are derived from the definition of S_\star and the second equality is derived from Lemma 2.2.

FIGURE 12. S_\star satisfies a relation similar to the STU relation

Hence S_{m+1} satisfies a relation similar to the STU relation; this means

$$s_\sigma - s_{(k \ k+1) \circ \sigma} = \begin{cases} s_{\hat{\sigma}} & \text{if } \sigma^{-1}(k) - \sigma^{-1}(k+1) = -1 \\ -s_{\hat{\sigma}} & \text{if } \sigma^{-1}(k) - \sigma^{-1}(k+1) = 1 \\ 0 & \text{if } |\sigma^{-1}(k) - \sigma^{-1}(k+1)| \geq 2 \end{cases}$$

where we define $\hat{\sigma}$ by putting $\hat{\sigma}^{-1}(j)$ to be $\sigma^{-1}(j)$ if $j \leq k$, $\sigma^{-1}(j+1) - 1$ if $j > k$; note that we can obtain the chord diagram $S_{\hat{\sigma}}$ from S_σ by gluing two adjacent dashed edges who have univalent vertices k and $k+1$ respectively, and we can define $\hat{\sigma}$ only when $\sigma^{-1}(k) - \sigma^{-1}(k+1) = \pm 1$. We note that we can use (2.1) for $\hat{\sigma}$ by hypothesis of induction. Since we can check that the above formula satisfies (2.1), we obtain the required claim of Step 2.

Since we can obtain any $\sigma \in \mathcal{S}_{m-1}$ from some permutation $(0, 1, 2, \dots, k)$ by composing interchanges $(j \ j+1)$, we can show (2.1) for any σ , completing the proof. \square

By Lemmas 2.3 and 2.4, we immediately obtain the following proposition.

Proposition 2.6. *The chord diagram T_m is symmetric with respect to the action of dihedral group of order $2m$; we show simple cases in Figure 13. Note that the sign of T_m possibly changes by the action as in the statements of Lemmas 2.3 and 2.4.*

$$\begin{aligned}
T_2 &= 0 \text{-----} 1 \\
T_3 &= \frac{1}{2} \begin{array}{c} 0 \text{-----} 2 \\ | \\ 1 \end{array} \\
T_4 &= \frac{1}{3} \begin{array}{c} 0 \text{-----} 3 \\ | \quad | \\ 1 \quad 2 \end{array} - \frac{1}{6} \begin{array}{c} 0 \text{-----} 3 \\ | \quad | \\ 2 \quad 1 \end{array} \\
&= \frac{1}{6} \begin{array}{c} 0 \quad 3 \\ \diagdown \quad \diagup \\ \text{-----} \\ \diagup \quad \diagdown \\ 1 \quad 2 \end{array} + \frac{1}{6} \begin{array}{c} 0 \quad 3 \\ \diagdown \quad \diagup \\ \text{-----} \\ \diagup \quad \diagdown \\ 1 \quad 2 \end{array}
\end{aligned}$$

FIGURE 13. Simple cases of T_m

Noting the symmetry of T_m in Proposition 2.6, we can obtain the following proposition from Lemma 2.2.

Proposition 2.7. *The difference between T_m and the chord diagram obtained by changing any adjacent two univalent vertices is equal to T_{m-1} with one extra trivalent vertex as shown in Figure 8.*

2.3. Replacing solid circles with dashed lines. Let n be a positive integer. For an integer m with $m \geq 2n$, we define $T_m^n \in \mathcal{A}(m)$ as follows. We divide m into n integers as;

$$m = m_1 + m_2 + \cdots + m_n, \quad m_1 \geq m_2 \geq \cdots \geq m_n \geq 2.$$

Let T_m^n be the sum of all configurations of disjoint union of T_{m_i} 's where we take the configurations preserving cyclic order of univalent vertices of each T_{m_i} . We show the picture of T_5^2 in Figure 14. If m is less than $2n$, then we put T_m^n to be 0.

Proposition 2.8. *For any positive integer n , the difference between T_m^n and the chord diagram obtained by changing any adjacent two univalent vertices is equal to T_{m-1}^n with*

$$T_5^2 = \frac{1}{2} (\text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{diagram 4} + \text{diagram 5} + \text{diagram 6} + \text{diagram 7} + \text{diagram 8} + \text{diagram 9} + \text{diagram 10})$$

FIGURE 14. The picture of T_5^2

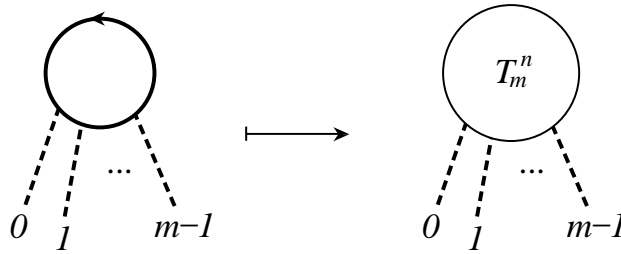
one extra trivalent vertex. Namely T_m^n also satisfies the same formula as in Figure 8 for T_m .

Proof. For each term in T_m^n , we consider the difference in the left hand side. If the two adjacent vertices are on different connected components of T_m^n , then the difference vanishes. If they are on the same connected component, then the difference is equal to a term in T_{m-1}^n in the right hand side by Proposition 2.7. Hence we obtain the required formula. \square

Using the above proposition, we can define a linear map

$$\iota_n : \mathcal{A}(\coprod^l S^1) \longrightarrow \mathcal{A}(\phi)$$

by putting ι_n (a solid circle with m dashed univalent vertices) to be T_m^n as shown in Figure 15. We also denote by the same notation ι_n the naturally extended map of $\hat{\mathcal{A}}(\coprod^l S^1)$ to $\hat{\mathcal{A}}(\phi)$.

FIGURE 15. The definition of ι_n

3. A SERIES OF INVARIANTS OF 3-MANIFOLDS

In this section we will construct a series of topological invariants $\Omega_n(M)$ of a 3-manifold M . We show the invariance of $\check{Z}(L)$ under orientation change and Kirby move II using the equivalence relation P_\star defined below. Since the relation P_{n+1} vanish in low degrees

in the image of ι_n , we will obtain a series of invariants in low degrees of $\mathcal{A}(\phi)$ through the maps ι_n .

3.1. Invariance under orientation change and Kirby move II. Let $\mathring{\mathcal{A}}(X)$ be the space of chord diagrams including dashed trivial circles. We denote by $\mathring{\mathcal{A}}(X)^\wedge$ its completion. For each positive integers n , we define an equivalence relation P_n in $\mathring{\mathcal{A}}(X)$ and $\mathring{\mathcal{A}}(X)^\wedge$ as follows. Let P_1 be the equivalence relation such that any chord diagram with non-empty dashed lines is equivalent to zero. Let P_2 be the equivalence relation shown in Figure 16; the left hand side of the first formula in Figure 16 is the sum over all pairings of 4 points. Similarly we define the equivalence relation P_n such that the sum over all pairings of $2n$ points is equivalent to zero.

$$\begin{array}{l}
 P_2 : \quad \text{[diagram 1]} + \text{[diagram 2]} + \text{[diagram 3]} = 0 \\
 P_3 : \quad \text{[diagram 4]} + \text{[diagram 5]} + \text{[diagram 6]} + \dots = 0
 \end{array}$$

FIGURE 16. The relations P_2 and P_3

Before proving the invariance of $\check{Z}(L)$ in Proposition 3.3, we prepare the following lemma.

Lemma 3.1. *Let D be a chord diagram on X , and C a component of X .*

1. *Then, with the equivalence relation P_{n+1} , the chord diagram D is equivalent to a linear sum of chord diagrams each of which has at most $2n$ univalent vertices on C .*
2. *Further, the chord diagram D becomes equivalent to a linear sum of chord diagrams each of which has either n isolated dashed chords on C or at most $2n - 1$ univalent vertices on C . Here we mean by an isolated chord a dashed arc with no trivalent vertices and two adjacent univalent vertices on C .*

Proof. We will see the case $n = 3$ before the general case. It is sufficient to show that, if the diagram D has k univalent vertices on C with $k > 4$, then it is equivalent to a linear sum of chord diagrams each of which has less than k univalent vertices; we will call them lower terms. We use the relation P_3 as in Figure 17, where the second equality is derived

from the STU relation; we can use the STU relation to interchange two univalent vertices modulo a lower term.

$$\begin{aligned}
 \text{Diagram } P_3 &= \text{Diagram 1} + \text{Diagram 2} + \dots \\
 &= 6 \text{Diagram 3} + 3 \text{Diagram 4} + 3 \text{Diagram 5} + 3 \text{Diagram 6} + (\text{lower terms}) = 0
 \end{aligned}$$

FIGURE 17. Using P_3 to make isolated chords

Then we can replace D with a chord diagram with an isolated chord, where we mean by an isolated chord a trivial dashed arc with adjacent univalent vertices on X . Iterating this procedure, we can replace D with a chord diagram with at least two isolated chords. Then we can use the relation P_3 again as in Figure 18, and we can replace the diagram with lower terms, completing the proof of (1) for the case $n = 3$.

$$\text{Diagram } P_3 = 15 \text{Diagram 1} + (\text{lower terms}) = 0$$

FIGURE 18. Using P_3 to decrease univalent vertices

In order to prove (2), it is sufficient to show that any chord diagram with 4 univalent vertices on C is equivalent to a chord diagram with two isolated chords on C modulo lower terms. We use the relation in Figure 17 again, to make one isolated chord. We further use the relation P_3 as in Figure 19. Then we can replace the diagram with a diagram with two isolated chords on C , completing the proof of (2) for the case $n = 3$.

$$\text{Diagram } P_3 = 3 \text{Diagram 1} + 12 \text{Diagram 2} + (\text{lower terms}) = 0$$

FIGURE 19. Using P_3 to make two isolated chords

In a general case for proving (1), it is sufficient to show that, if the diagram D has m univalent vertices on C with $m > 2n$, then it is equivalent to a linear sum of chord diagrams with at most $m - 1$ univalent vertices. We use the relation P_{n+1} as in Figure 20 for $0 \leq k \leq n$; we use it for $k = n$ at the beginning to make an isolated chord, and use it for $k = 2, 3, \dots$ to increase isolated chords, and finally use it for $k = 0$ to replace the diagram with lower terms; note that we can do it assuming $m > 2n$. This completes the proof of (1).

In order to prove (2), it is sufficient to show that any chord diagram with $2n$ univalent vertices on C is equivalent to a diagram with n isolated chords on C modulo lower terms. We use the relation in Figure 20 as above to make $n - 1$ isolated chords on C . We further use the relation for $k = 2$, and we obtain a chord diagram with n isolated chords on C . This completes the proof of (2). \square

$$P_{n+1} = a_{k+l} \text{ (} n-k \text{ isolated chords)} + a_{k-l} \text{ (} n-k+l \text{ isolated chords)} + \dots + a_l \text{ (} n-k/2 \text{ isolated chords)} + \text{(lower terms)} = 0$$

FIGURE 20. Using P_{n+1} to decrease univalent vertices; this picture is for even k

Remark 3.2. We can uniquely characterize T_m^n as an element of $\mathcal{A}(m)/P_{n+1}$ by the following four conditions.

1. It satisfies $T_m^n = 0$ if $m < 2n$.
2. It is invariant under cyclic permutation of the m external univalent vertices.
3. It satisfies a relation similar to the STU relation; in other words, satisfies Proposition 2.8.
4. It contains neither dashed cycle nor dashed component; that is, it is equal to a linear sum of chord diagrams each of which is a disjoint union of simply connected chord diagrams having external vertices.

Outline of a proof of the fact is as follows. By Lemma 3.1 a solid circle with m dashed chords is equivalent (modulo P_{n+1} and lower terms) to the disjoint union of a solid circle with n isolated dashed chords and a linear sum of dashed trivalent graphs. We can obtain

an alternative definition of ι_n ; we define the inverse map ι_n^{-1} by removing the solid circle with n dashed isolated chords from the above disjoint union. The image of solid circle with m short dashed chords through ι_n^{-1} is equal to T_m^n by definition of ι_n , and we can check the image satisfies the above four conditions.

By Lemma 3.1 we can replace a solid circle with a solid circle with $2n$ univalent vertices modulo lower terms. In the following proposition we show the invariance of $\check{Z}(L)$ by reducing the proof to the above particular solid circle ignoring the lower terms.

Proposition 3.3. *Let L be any oriented framed link of l components, n any positive integer.*

1. *The equivalence class $[\check{Z}(L)]$ including $\check{Z}(L)$ in $\mathring{\mathcal{A}}(\coprod^l S^1)^\wedge / L_{<2n}, P_{n+1}, O_n$ does not depend on orientation of L , where we denote by $L_{<2n}$ the equivalence relation such that any chord diagram including a solid circle with less than $2n$ dashed univalent vertices is equivalent to zero, and O_n the equivalence relation such that a trivial dashed circle is equivalent to $-2n$.*
2. *The above equivalence class is invariant under Kirby move II.*

Proof. We will show the proposition for each chord diagram D composing $\check{Z}(L)$.

Let L' be the framed link obtained from L by changing the orientation of a component C ; we denote by C' the corresponding solid circle in the chord diagram in $\check{Z}(L)$. The change of $\check{Z}(L)$ to $\check{Z}(L')$ is given in (1.3); the diagram in $\check{Z}(L')$ obtained by changing D is equal to $S_{(C')}(D)$. We will show that $S_{(C')}(D)$ is equivalent to D .

If D has less than $2n$ dashed univalent vertices on C' , then both of D and $S_{(C')}(D)$ vanish by the relation $L_{<2n}$. Hence they are equivalent.

If D has $2n$ univalent vertices on C' , then by (1.3) we can obtain $S_{(C')}(D)$ from D by changing the order of univalent vertices on C' . By the relation $L_{<2n}$ and the STU relation, their equivalence classes are invariant under changing the order of univalent vertices on C' in this case. Hence the equivalence classes are equivalent.

If D has k univalent vertices on C' for $k > 2n$, then we can reduce this case to the case $k = 2n$ as follows. By Lemma 3.1 we can replace D with a chord diagram with less univalent vertices on C . The relations we used in the proof are only the STU relation and the relation P_{n+1} . Namely, we have a series of a linear sum of chord diagrams; $D = D_0, D_1, D_2, \dots, D_N$, where D_i and D_{i+1} are related by either of the STU relation or the relation P_{n+1} . Note that, if D_i and D_{i+1} are related by the STU relation, then

$S_{(C)}(D_i)$ and $S_{(C)}(D_{i+1})$ are also related by the relation, because $S_{(C)}$ and the STU relation commute as shown in Figure 21; in fact, it guarantees that the map $S_{(C)}$ is well defined. Further note that the relation P_{n+1} and the STU relation commute, because the relation P_{n+1} is independent of solid chords. Hence $S_{(C)}(D_i)$ and $S_{(C)}(D_{i+1})$ are related by the STU relation or the relation P_{n+1} . Therefore, if D_{i+1} is equivalent to $S_{(C)}(D_{i+1})$, then if D_i is equivalent to $S_{(C)}(D_i)$; this implies the required reduction, completing this case. This completes the proof of (1).

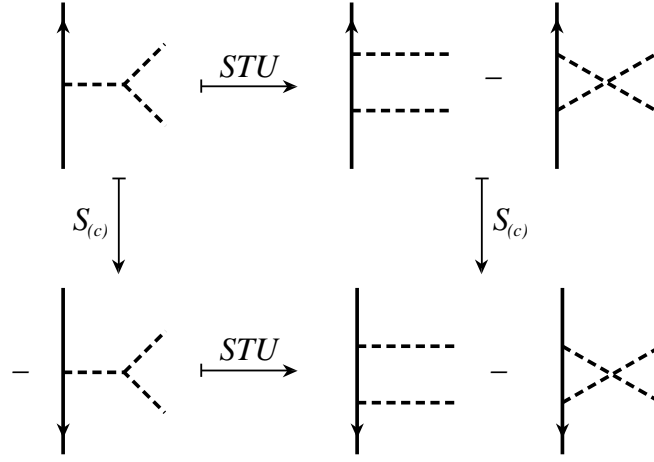


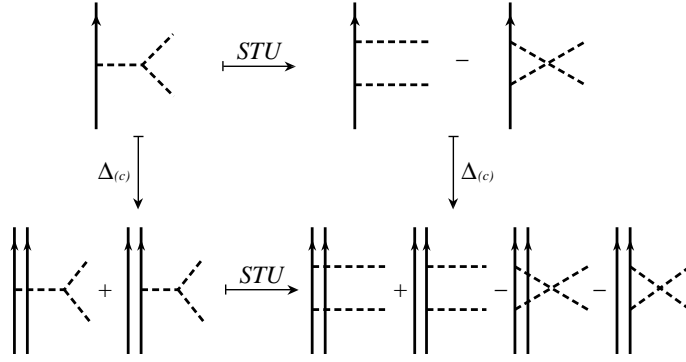
FIGURE 21. The map $S_{(C)}$ and the STU relation commute

Let L'' be a framed link obtained from L by Kirby move II taking handle sliding of some component over a component C . The change of $\check{Z}(L)$ to $\check{Z}(L'')$ is given by Proposition 1.3; the diagram in $\check{Z}(L'')$ obtained by changing D is equal to $\Delta_{(C)}(D)$. We will show that $\Delta_{(C)}(D)$ is equivalent to D .

If there are at most $2n - 1$ univalent vertices on C' , then both of D and $\Delta_{(C)}(D)$ vanish by the relation $L_{<2n}$. Hence they are equivalent.

If there are $2n$ univalent vertices on C' , then by Proposition 1.3 $\Delta_{(C)}(D)$ is equal to a sum of D and the other terms. Further there are less than $2n$ univalent vertices on C' for each of the other terms. Hence their equivalent classes vanish by the relation $L_{<2n}$. Therefore D and $\Delta_{(C)}(D)$ are equivalent.

If there are k univalent vertices on C' for $k > 2n$, then we can reduce this case to the case $k - 1$ as in the above proof of (1). Instead of the commutation of $S_{(C)}$ and the STU relation, we need the commutation of $\Delta_{(C)}$ and the STU relation in the present case; we show it in Figure 22.

FIGURE 22. The map $\Delta_{(C)}$ and the STU relation commute

This completes the proof of Proposition 3.3. \square

3.2. Moving $\check{Z}(L)$ into a set with algebra structure. In order to show the invariance under Kirby move I, we move $\check{Z}(L)$ into a quotient space of $\mathcal{A}(\phi)$, in which there is an algebra structure with respect to the disjoint union of chord diagrams. Before moving, we prepare the following lemma, which guarantees that we need not consider P_{n+1} in low degrees of $\mathcal{A}(\phi)$.

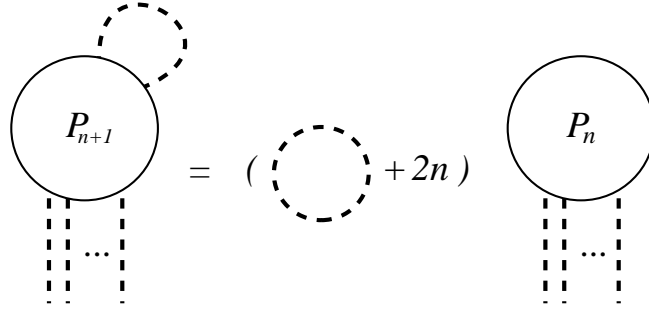
Lemma 3.4. *The identity map induces an isomorphism of the quotient space $\mathcal{A}(\phi)/D_{>n}$ to the quotient space $\mathring{\mathcal{A}}(\phi)/D_{>n}, P_{n+1}, O_n$.*

Proof. We can remove the trivial dashed component in $\mathring{\mathcal{A}}(\phi)$ by O_n . Hence it is sufficient to show the claim that, if an element of $\mathring{\mathcal{A}}(\phi)$ including P_{n+1} , then either it vanishes or its degree is greater than n , where we also denote by P_{n+1} the formula defining the relation P_{n+1} . Note that P_{n+1} is symmetric with respect to the action of \mathcal{S}_{2n} acting on the set of $2n$ ends of P_{n+1} . We will show the claim for any outside of P_{n+1} .

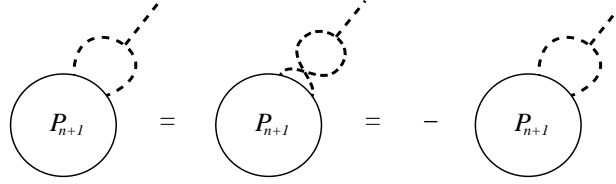
We will show that the degree must be more than n unless the element vanishes. We can make an injection of the set of $2n + 2$ ends of P_{n+1} to the set of trivalent vertices in the outside as follows. Choose an end of P_{n+1} .

If the end is not connected to any other end in the outside, we associate one of trivalent vertices in the connected component of the end in the outside. Otherwise we can find a path connecting the end to one of the other ends in the outside.

If there are no trivalent vertices on the path, the element vanishes with the relation O_n as shown in Figure 23.

FIGURE 23. The relation P_{n+1} vanishes

If there are one trivalent vertices on the path, the element vanishes using the symmetry of P_{n+1} and the AS relation as shown in Figure 24.

FIGURE 24. The relation P_{n+1} vanishes again

Otherwise there must be at least two trivalent vertices on the path. Then we associate the nearest trivalent vertex to the end, to obtain an injection of the set of $2n + 2$ ends of P_{n+1} to the set of trivalent vertices.

Hence we showed that the number of trivalent vertices is greater than $2n$; this implies that the degree is greater than n , completing the proof. \square

Proposition 3.5. 1. *The equivalence class $[\iota_n(\check{Z}(L))] \in \mathcal{A}(\phi)/D_{>n}$ does not depend on orientation of L .*

2. *Further, the above equivalence class does not change under Kirby move II.*

Proof. By the map ι_n any chord diagram with less than $2n$ univalent vertices on a solid circle of the diagram vanishes. Hence we have the following map;

$$\mathring{\mathcal{A}}(\coprod^l S^1)^\wedge / L_{<2n}, P_{n+1}, O_n \xrightarrow{\iota_n} \mathring{\mathcal{A}}(\phi)^\wedge / P_{n+1}, O_n \xrightarrow{\text{proj}} \mathring{\mathcal{A}}(\phi) / D_{>n}, P_{n+1}, O_n$$

where the first map is the map induced by ι_n which we also denote by ι_n , and the second map is the projection. By Proposition 3.3 the equivalence class $[\check{Z}(L)]$ in the first set

$\mathring{\mathcal{A}}(\coprod^l S^1)^\wedge / L_{<2n}, P_{n+1}, O_n$ is invariant under both of orientation change and Kirby move II. Hence it is also true for the equivalence class $[\iota_n \check{Z}(L)]$ in the third set $\mathring{\mathcal{A}}(\phi) / D_{>n}, P_{n+1}, O_n$, which is naturally isomorphic to the set $\mathcal{A}(\phi) / D_{>n}$ by Lemma 3.4. This completes the proof. \square

Using the map $\hat{\Delta}$ we will show a proof of the following lemma in the following section.

Lemma 3.6. *Let U_+ (resp. U_-) be the trivial knot with $+1$ (resp. -1) framing. Then $[\iota_n(\check{Z}(U_\pm))]$ is invertible in $\mathcal{A}(\phi) / D_{>n}$, in which we define an algebra structure such that the disjoint union of two chord diagrams is the product of the diagrams.*

Using the above proposition and lemma, we obtain the following theorem.

Theorem 3.7. *Let L be an oriented framed link, and M the 3-manifold obtained by Dehn surgery on S^3 along L . Then the equivalence class*

$$[\iota_n(\check{Z}(U_+))^{-\sigma_+} [\iota_n(\check{Z}(U_-))^{-\sigma_-} [\iota_n(\check{Z}(L))]] \in \mathcal{A}(\phi) / D_{>n}$$

is a topological invariant of M for any positive integer n , where we denote by σ_+ (resp. σ_-) the number of positive (resp. negative) eigenvalues of the linking matrix of L .

Proof. We obtain invariance under both of orientation change of L and Kirby move II by Proposition 3.5. Note that we can apply Kirby move II in any way though we prove the proposition for Kirby move II preserving the orientation of L , because we also showed the invariance under orientation change.

We also obtain invariance under Kirby move I, since the change of $[\iota_n \check{Z}(L)]$ under the move cancels with the change of σ_\pm . \square

Definition 3.8. We denote the above invariant by $\Omega_n(M)$.

4. A UNIVERSAL QUANTUM INVARIANT OF 3-MANIFOLDS

In this section we unify the series $\Omega_n(M)$ into a invariant $\Omega(M)$. We show that the series $\Omega_n(M)$ satisfies a property, and that by the property the series has the same information (modulo the order of the first homology group) as $\Omega(M)$ has. We further show that the invariant $\Omega(M)$ satisfies a property derived from the above property of the series, and that there exists the logarithm of $\Omega(M)$ by the property; we denote by the logarithm a universal quantum invariant of 3-manifolds.

4.1. A group-like property of the series $\Omega_n(M)$. We denote by $\hat{\Delta}_{n_1, n_2}$ the map $\mathcal{A}(\phi)/D_{>n_1+n_2} \rightarrow \mathcal{A}(\phi)/D_{>n_1} \otimes \mathcal{A}(\phi)/D_{>n_2}$ naturally induced by $\hat{\Delta}$; note that this map is well defined since the degree is preserved by $\hat{\Delta}$, where we regard the sum of degrees as the degree in the tensor product. Using Theorem 1.2 we obtain the following proposition.

Proposition 4.1.

$$\hat{\Delta}_{n_1, n_2}(\Omega_{n_1+n_2}(M)) = \Omega_{n_1}(M) \otimes \Omega_{n_2}(M).$$

Proof. Noting that $\hat{\Delta}$ is an algebra homomorphism in $\mathcal{A}(\phi)$, this proposition is a direct conclusion of Theorem 1.2 and the following lemma. \square

Lemma 4.2. *Let n , n_1 and n_2 be positive integers satisfying $n = n_1 + n_2$. Then the following diagram is commutative;*

$$\begin{array}{ccc} \mathring{\mathcal{A}}(\amalg^l S^1)^\wedge / L_{<2n}, PO_n & \xrightarrow{\iota_n} & \mathring{\mathcal{A}}(\phi)^\wedge / PO_n \\ \hat{\Delta}_{n_1, n_2} \downarrow & & \downarrow \hat{\Delta} \\ \mathring{\mathcal{A}}(\amalg^l S^1)^\wedge / L_{<2n_1}, PO_{n_1} \otimes \mathring{\mathcal{A}}(\amalg^l S^1)^\wedge / L_{<2n_2}, PO_{n_2} & \xrightarrow{\iota_{n_1} \otimes \iota_{n_2}} & \mathring{\mathcal{A}}(\phi)^\wedge / PO_{n_1} \otimes \mathring{\mathcal{A}}(\phi)^\wedge / PO_{n_2} \end{array} \quad (4.1)$$

where we mean by PO_k the equivalence relation generated by P_{k+1} and O_k .

Proof. By Lemma 3.1 we can assume that an element in $\mathring{\mathcal{A}}(\amalg^l S^1)^\wedge / L_{<2n}, PO_n$ is a disjoint union of dashed trivalent graphs and l copies of a solid circle with n dashed isolated chords; in fact, the space consists of linear sums of such elements. Since ι_\star is trivial for dashed trivalent graphs, (4.1) is commutative for them. Hence it is sufficient to show that (4.1) is commutative for a solid circle with n dashed isolated chords; we put it to be D_n .

The image of D_n by the map ι_n is equal to the chord diagram obtained by attaching n isolated chords to T_{2n}^n , by the definition of ι_n . By the same argument in Figure 23, T_{2n}^n with one dashed isolated chords is equal to T_{2n-2}^{n-1} times a sum of $2n-2$ and a dashed circle; the sum is equal to -2 with the relation O_n . Repeating this argument, we can show that the image is equal to $(-2)(-4)\cdots(-2n) = (-2)^n n!$. Therefore the clockwise image of D_n in the diagram is equal to $(-2)^n n!$.

The image of D_n by the map $\hat{\Delta}$ is equal to the sum of $\binom{n}{k} D_k \otimes D_{n-k}$ by the definition of $\hat{\Delta}$. Note that it vanishes with $L_{<2n_1}$ or $L_{<2n_2}$ unless $k = n_1$. Hence the image by $\hat{\Delta}_{n_1, n_2}$ is equal to $\binom{n}{n_1} D_{n_1} \otimes D_{n_2}$. By the same argument as above, the image of it by the map $\iota_{n_1} \otimes \iota_{n_2}$ is equal to $\binom{n}{n_1} (-2)^{n_1} n_1! (-2)^{n_2} n_2! = (-2)^n n!$ using $n = n_1 + n_2$; this coincides the above value. \square

Proof of Lemma 3.6. Applying Theorem 1.2 to $\check{Z}(U_{\pm})$, $n - 1$ times, we have the formula $\hat{\Delta}^{(n-1)}(\check{Z}(U_{\pm})) = (\check{Z}(U_{\pm}))^{\otimes n}$, where we define the map $\hat{\Delta}^{(k)} : \hat{\mathcal{A}}(X) \rightarrow \hat{\mathcal{A}}(X)^{\otimes(k+1)}$ by $\hat{\Delta}^{(k)} = (\hat{\Delta} \otimes 1) \circ \hat{\Delta}^{(k-1)}$ recursively. Further, putting $X = \phi$, the map $\hat{\Delta}^{(k)}$ naturally induces the map of $\mathcal{A}(\phi)/D_{>k+1}$ to $(\mathcal{A}(\phi)/D_{>1})^{\otimes(k+1)}$; we denote it by $\hat{\Delta}_{1,1,\dots,1}^{(k)}$. Applying Lemma 4.2 to the above formula $n - 1$ times, we obtain the first equality of the following formula;

$$\hat{\Delta}_{1,1,\dots,1}^{(n-1)}([\iota_n(\check{Z}(U_{\pm}))]) = (\iota_1(\check{Z}(U_{\pm})))^{\otimes n} = (\mp 1 + \frac{\theta}{16})^{\otimes n} \in (\mathcal{A}(\phi)/D_{>1})^{\otimes n}.$$

Here the second equality is derived from Lemma 4.3 below. Therefore the constant term of $[\iota_n(\check{Z}(U_{\pm}))]$ does not vanish, which means that it is invertible. \square

We have the following lemma proved in [13].

Lemma 4.3 ([13]). *The following formula holds;*

$$[\iota_1(\check{Z}(U_{\pm}))] = \mp 1 + \frac{\theta}{16} \in \mathcal{A}(\phi)/D_{>1}$$

where we mean by θ the dashed trivalent graph consisting of three edges and two vertices, as the Greek character θ .

Remark 4.4. In fact, the above values of $\check{Z}(U_{\pm})$ are -2 times the values in [13]. The difference occurs from the definition of the map ι_1 ; it was defined by simply removing Θ components in [13], where we mean by Θ a solid circle with one dashed line as the Greek letter Θ . On the other hand, we define ι_1 here in the way how we replace Θ with the trivial dashed circle, which is equivalent to -2 .

4.2. A power series invariant $\Omega(M)$. We define a map $\varepsilon : \mathcal{A}(\phi) \rightarrow \mathbb{C}$ to be the projection to the degree 0 part of a linear sum of chord diagrams; recall that we regard the empty diagram as 1 whose degree is 0. By the definitions of $\hat{\Delta}_{1,n-1}$ and ε , we immediately obtain the following lemma.

Lemma 4.5. *The following formula holds;*

$$(\varepsilon \otimes 1) \circ \hat{\Delta}_{1,n-1} = p_{n,n-1}$$

where we mean by 1 the identity map and we denote by $p_{n,n-1}$ the projection of $\mathcal{A}(\phi)/D_{>n}$ to $\mathcal{A}(\phi)/D_{>n-1}$.

Lemma 4.6. *Let $(\Omega_1, \Omega_2, \dots)$ be a series of $\Omega_n \in \mathcal{A}(\phi)/D_{>n}$ which satisfies*

$$\hat{\Delta}_{n_1, n_2}(\Omega_{n_1+n_2}) = \Omega_{n_1} \otimes \Omega_{n_2} \quad (4.2)$$

for any positive integers n_1 and n_2 .

1. *Then the formula $\Omega_n^{(d)} = m^{n-d} \Omega_d^{(d)}$ holds for $d < n$, where we denote by $\alpha^{(d)}$ the degree d part of α and we put m to be $\Omega_1^{(0)}$.*
2. *Further, for $\Omega \in \hat{\mathcal{A}}(\phi)$ defined to be $1 + \sum_{n=1}^{\infty} \Omega_n^{(n)}$, the formula $\hat{\Delta}(\Omega) = \Omega \otimes \Omega$ holds; recall that we denote by $\hat{\mathcal{A}}(\phi)$ the completion of $\mathcal{A}(\phi)$ with respect to the degree, that is, $\hat{\mathcal{A}}(\phi)$ consists of infinite linear sums of chord diagrams. We also denote by $\hat{\Delta}$ the natural extension of $\hat{\Delta}$ to $\hat{\mathcal{A}}(\phi)$.*

Proof. We apply the map in Lemma 4.5 to Ω_n . Then we have the left hand side as;

$$(\varepsilon \otimes 1) \circ \hat{\Delta}_{1, n-1}(\Omega_n) = (\varepsilon \otimes 1)(\Omega_1 \otimes \Omega_{n-1}) = m \Omega_{n-1}.$$

Hence we obtain (1).

We put Ω_0 to be 1, then the series $(\Omega_0, \Omega_1, \dots)$ satisfies (4.2) for any non-negative integers n_1 and n_2 . It is sufficient to show the required formula of (2) in each degree n . Hence we will show the formula $\hat{\Delta}(\Omega_n^{(n)}) = \sum_{k_1+k_2=n} \Omega^{k_1} \otimes \Omega^{k_2}$. Though this is a formula in $\hat{\mathcal{A}}(\phi) \otimes \hat{\mathcal{A}}(\phi)$, it consists only of degree n part. Therefore it suffices to show it for the image of it by the map $p_{\infty, n_1} \otimes p_{\infty, n_2}$ for each pair n_1 and n_2 with $n_1 + n_2 = n$, where we denote by $p_{\infty, k}$ the projection of $\hat{\mathcal{A}}(\phi)$ to $\mathcal{A}(\phi)/D_{>k}$. The image becomes $\hat{\Delta}_{n_1, n_2}(\Omega_n^{(n)}) = \Omega_{n_1}^{(n_1)} \otimes \Omega_{n_2}^{(n_2)}$ which is a special case of (4.2). This completes the proof. \square

We have the following lemma by results in [12] where only the invariant $\Omega_1(M)$ was discussed.

Lemma 4.7. *The degree zero part of $\Omega_1(M)$ is equal to $|H_1(M, \mathbb{Z})|$ if M is a rational homology 3-sphere, 0 otherwise. Here we mean by $|\cdot|$ the order of the set.*

We know that we can apply Lemma 4.6 to our series of invariants $\Omega_n(M)$ by Proposition 4.1, to reduce the series to $\Omega \in \hat{\mathcal{A}}(\phi)$ and a scalar m . In our case the scalar m becomes either the order of the first homology group or zero by Lemma 4.7. Then we obtain the following definition and proposition from Lemma 4.6 (2).

Definition 4.8. We define a topological invariant $\Omega(M)$ of a 3-manifold M by $\Omega(M) = 1 + \sum_{n=1}^{\infty} \Omega_n(M)^{(n)} \in \hat{\mathcal{A}}(\phi)$.

Proposition 4.9. *The invariant $\Omega(M)$ satisfies $\hat{\Delta}(\Omega(M)) = \Omega(M) \otimes \Omega(M)$.*

4.3. Logarithm of $\Omega(M)$. We denote by $\mathcal{A}(\phi)_{\text{conn}}$ the vector subspace of $\mathcal{A}(\phi)$ spanned by the set of connected non-empty dashed trivalent graphs with oriented trivalent vertices subject to AS and IHX relations. We put $\hat{\mathcal{A}}(\phi)_{\text{conn}}$ to be the completion of it with respect to the degree, which becomes a vector subspace of $\hat{\mathcal{A}}(\phi)$.

It is well known as a property of Hopf algebra, see for example [1], that a non-zero element $\Omega \in \hat{\mathcal{A}}(\phi)$ is group-like if and only if there exists a primitive element $\omega \in \hat{\mathcal{A}}(\phi)$ satisfying $\Omega = \exp(\omega) = 1 + \omega + (1/2)\omega^2 + \dots$, where we call α *group-like* if it satisfies $\hat{\Delta}(\alpha) = \alpha \otimes \alpha$, and call α *primitive* if it satisfies $\hat{\Delta}(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha$. In our case $\alpha \in \hat{\mathcal{A}}(\phi)$ is primitive if and only if α belongs to $\hat{\mathcal{A}}(\phi)_{\text{conn}}$. Note that ω is uniquely determined for given Ω , since a primitive element always has a positive degree.

Since $\Omega(M)$ is group-like by Proposition 4.9, we have the following definition.

Definition 4.10. We define $\omega(M) \in \hat{\mathcal{A}}(\phi)_{\text{conn}}$ by $\exp(\omega(M)) = \Omega(M)$. We call $\omega(M)$ a universal quantum invariant.

5. PROPERTIES OF THE UNIVERSAL QUANTUM INVARIANT $\omega(M)$

We will see some properties of $\omega(M)$ in this section.

5.1. Formulas for connected sum and opposite orientation.

Proposition 5.1. *Let M be the connected sum of two closed 3-manifolds M_1 and M_2 , then $\omega(M)$ is given by*

$$\omega(M) = \sum_{d=1}^{\infty} \left(m_2^d \omega(M_1)^{(d)} + m_1^d \omega(M_2)^{(d)} \right),$$

where we put m_i ($i = 1, 2$) to be the order of $H_1(M_i, \mathbb{Z})$ if M_i is a rational homology 3-sphere, 0 otherwise.

Proof. Let L be the framed link obtained by taking split union of two framed links L_1 and L_2 . By definition of $\check{Z}(L)$ we have $\check{Z}(L)$ as the disjoint union of $\check{Z}(L_1)$ and $\check{Z}(L_2)$.

Note that, if M_1 and M_2 are obtained by Dehn surgery along L_1 and L_2 respectively, then M is obtained from L . By definition of $\Omega_n(M)$ we have $\Omega_n(M) = \Omega_n(M_1)\Omega_n(M_2)$, recall that we define multiplication by disjoint union of chord diagrams. By Lemmas 4.6

and 4.7 we have

$$\begin{aligned}\Omega_n(M)^{(n)} &= \sum_{d_1+d_2=n} \Omega_n(M_1)^{(d_1)} \Omega_n(M_2)^{(d_2)} \\ &= \sum_{d_1+d_2=n} m_1^{d_2} \Omega_{d_1}(M_1)^{(d_1)} m_2^{d_1} \Omega_{d_2}(M_1)^{(d_2)}.\end{aligned}$$

Hence we have

$$\Omega(M) = \sum_{d_1, d_2=0}^{\infty} m_1^{d_2} \Omega(M_1)^{(d_1)} m_2^{d_1} \Omega(M_2)^{(d_2)}.$$

If both of M_1 and M_2 are rational homology 3-spheres, the above formula implies that $\sum_d \Omega(M_i)^{(d)} / m_i^d$ is multiplicative with respect to connected sum. Hence $\sum_d \omega(M_i)^{(d)} / m_i^d$ is additive, and we obtain the required formula.

If either of M_1 and M_2 , say M_1 , is not a rational homology 3-sphere, then we have $m_1 = 0$. Hence $\Omega(M) = \sum_d \Omega(M_1)^{(d)} m_2^d$ by the above formula putting $d_2 = 0$, where we regard 0^0 as 1. Therefore we obtain the required formula, completing the proof. \square

Proposition 5.2. *Let $-M$ be a 3-manifold M with opposite orientation. Then the following formula holds;*

$$\omega(-M) = \sum_{d=1}^{\infty} (-1)^d \omega(M)^{(d)}.$$

Proof. We define a map $\hat{S} : \mathcal{A}(X) \rightarrow \mathcal{A}(X)$ by putting $\hat{S}(D) = (-1)^d D$ for a chord diagram D where d is the degree of D . For the mirror image \bar{L} of a framed link L , we have $\check{Z}(\bar{L}) = \hat{S}(\check{Z}(L))$; we can show the formula by checking it for each elementary tangles. Since $-M$ is obtained by Dehn surgery along \bar{L} , we have $\Omega_n(-M) = \hat{S}(\Omega_n(M))$, $\Omega(-M) = \hat{S}(\Omega(M))$ and $\omega(-M) = \hat{S}(\omega(M))$ by their definitions. The last formula is the required one. \square

5.2. The first term in $\omega(M)$. By results in [13] we have the following proposition.

Proposition 5.3. *Let M be a oriented closed 3-manifold. Then the coefficient of dashed θ -curve in $\omega(M)$ is equal to $(-1)^{b_1(M)+1} 3\tilde{\lambda}(M)$ where $b_1(M)$ is the first Betti number of M and we denote $\tilde{\lambda}(M)$ twice Lescop's generalization [15] of the Casson-Walker invariant (Walker's normalization, which is twice Casson's normalization) $\lambda(M)$ [2, 21] satisfying $\tilde{\lambda}(M) = |H_1(M; \mathbb{Z})| \lambda(M)$ if M is a rational homology 3-sphere.*

Proof. In [13] it is shown that the degree 1 part of $\Omega_1(M)$ is equal to $(-1)^{b_1(M)+1} 3\tilde{\lambda}(M)$. The proposition immediately follows from that. \square

Remark 5.4. We can “substitute” a Lie algebra into dashed lines, see [5]. When we substitute sl_N , so_N and sp_N to dashed θ -curve, we obtain values $2N(N^2 - 1)$, $N(N - 1)(N - 2)/2$ and $2N(N + 1)(2N + 1)$ respectively. When we substitute sl_2 into the first term (i.e. dashed θ -curve and its coefficient) of $\omega(M)$, we have the value 12 times the coefficient. Hence, if one expects the existence of “ G Casson invariant $\lambda^G(M)$ ” and expects that it should recover from the first term in $\omega(M)$ by substituting the Lie algebra of G into dashed lines, it must satisfy

$$\begin{aligned}\lambda^{SU(N)}(M) &= \frac{N(N^2 - 1)}{6} \lambda(M), \\ \lambda^{SO(N)}(M) &= \frac{N(N - 1)(N - 2)}{24} \lambda(M), \\ \lambda^{Sp(N)}(M) &= \frac{N(N + 1)(2N + 1)}{6} \lambda(M).\end{aligned}$$

In [16] we have the same formula for $\lambda^{SU(N)}(M)$ of each lens space M , which is obtained expanding quantum $PSU(N)$ invariant of lens spaces obtained in [20] into a power series in $q - 1$.

REFERENCES

- [1] E. Abe, Hopf algebras, Cambridge University Press, 1980.
- [2] S. Akbulut and J. D. McCarthy, *Casson’s invariant for oriented homology 3-spheres, An exposition*, Mathematical Notes **36**, Princeton University Press, 1990.
- [3] S. Axelrod and I. M. Singer, *Chern-Simons perturbation theory*, Proc. XXth DGM Conference (New York, 1991) World Scientific, 3–45 (1992).
- [4] ———, *Perturbative aspects of Chern-Simons topological quantum field theory II*, J. Diff. Geom. **39**, 173–213 (1994).
- [5] D. Bar-Natan, *Weights of Feynman diagrams and the Vassiliev knot invariants*, preprint, 1991.
- [6] ———, *On the Vassiliev knot invariants*, Topology, **34**, 423–472 (1995).
- [7] R.C. Kirby, *A calculus for framed links in S^3* , Invent. Math. **65**, 35–56 (1978).
- [8] M. Kontsevich, *Vassiliev’s knot invariants*, Advances in Soviet Mathematics, **16**, 137–150 (1993).
- [9] ———, *Feynman diagrams and low-dimensional topology*, preprint.
- [10] T. Q. T. Le and J. Murakami, *Representations of the category of tangles by Kontsevich’s iterated integral*, Commun. Math. Phys., **168**, 535–562 (1995).
- [11] ———, *Parallel version of the universal Vassiliev-Kontsevich invariant*, preprint.
- [12] T. Q. T. Le, H. Murakami, J. Murakami and T. Ohtsuki, *A three-manifold invariant derived from the universal Vassiliev-Kontsevich invariant*, Proc. Japan Acad., **71**, Ser. A, 125–127 (1995).
- [13] ———, *A three-manifold invariant via the Kontsevich integral*, preprint, MPI/95-62.
- [14] T. Q. T. Le, J. Murakami and T. Ohtsuki, *An invariant of 3-manifolds derived from the universal Vassiliev-Kontsevich invariant*, preprint, 1995.
- [15] C. Lescop, *Sur l’invariant de Casson-Walker: formule de chirurgie globale et generalisation aux varietes de dimension 3 fermees orientees*, C. R. Acad. Sci. Paris Ser. I, **315**, 437–440 (1992).
- [16] T. Ohtsuki, *A polynomial invariant of rational homology 3-spheres*, to appear in Invent. Math.
- [17] N. Reshetikhin and V. G. Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups*, Invent. Math. **103**, 547–597 (1991).
- [18] L. Rozansky, *The trivial connection contribution to Witten’s invariant and finite type invariants of rational homology spheres*, preprint, 1995.

- [19] ———, *Witten's invariants of rational homology spheres at prime values of K and trivial connection contribution*, preprint, 1995.
- [20] T. Takata, *On Witten $SU(n)$ invariants for lens spaces*, preprint.
- [21] K. Walker, *An extension of Casson's invariant*, Annals of Math. Studies **126**, Princeton University Press, 1992.
- [22] E. Witten, *Quantum field theory and the Jones polynomial*, Commun. Math. Phys. **121**, 360–379 (1989).

